Therefore, $\cot^{-1}\dfrac{1}{\sqrt{x^2-1}} = \cot^{-1}(\cot\theta) = \theta = \sec^{-1} x$, which is the simplest form.

Example 7 Prove that $\tan^{-1} x + \tan^{-1}\dfrac{2x}{1-x^2} = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right),\ |x| < \dfrac{1}{\sqrt{3}}$

Solution Let $x = \tan\theta$. Then $\theta = \tan^{-1} x$. We have

$$\text{R.H.S.} = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1} x = \tan^{-1} x + 2\tan^{-1} x$$

$$= \tan^{-1} x + \tan^{-1}\frac{2x}{1-x^2} = \text{L.H.S. (Why?)}$$

Example 8 Find the value of $\cos(\sec^{-1} x + \operatorname{cosec}^{-1} x),\ |x| \geq 1$

Solution We have $\cos(\sec^{-1} x + \operatorname{cosec}^{-1} x) = \cos\left(\dfrac{\pi}{2}\right) = 0$

EXERCISE 2.2

Prove the following:

1. $3\sin^{-1} x = \sin^{-1}(3x - 4x^3),\ x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$
2. $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x),\ x \in \left[\dfrac{1}{2}, 1\right]$
3. $\tan^{-1}\dfrac{2}{11} + \tan^{-1}\dfrac{7}{24} = \tan^{-1}\dfrac{1}{2}$
4. $2\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{7} = \tan^{-1}\dfrac{31}{17}$

Write the following functions in the simplest form:

5. $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x},\ x \neq 0$
6. $\tan^{-1}\dfrac{1}{\sqrt{x^2-1}},\ |x| > 1$
7. $\tan^{-1}\left(\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right),\ x < \pi$
8. $\tan^{-1}\left(\dfrac{\cos x-\sin x}{\cos x+\sin x}\right),\ x < \pi$