Example 30 Show that $A \cup B = A \cap B$ implies $A = B$

Solution Let $a \in A$. Then $a \in A \cup B$. Since $A \cup B = A \cap B$, $a \in A \cap B$. So $a \in B$. Therefore, $A \subset B$. Similarly, if $b \in B$, then $b \in A \cup B$. Since

$A \cup B = A \cap B$, $b \in A \cap B$. So, $b \in A$. Therefore, $B \subset A$. Thus, $A = B$

Example 31 For any sets A and B, show that

$$P(A \cap B) = P(A) \cap P(B).$$

Solution Let $X \in P(A \cap B)$. Then $X \subset A \cap B$. So, $X \subset A$ and $X \subset B$. Therefore, $X \in P(A)$ and $X \in P(B)$ which implies $X \in P(A) \cap P(B)$. This gives $P(A \cap B) \subset P(A) \cap P(B)$. Let $Y \in P(A) \cap P(B)$. Then $Y \in P(A)$ and $Y \in P(B)$. So, $Y \subset A$ and $Y \subset B$. Therefore, $Y \subset A \cap B$, which implies $Y \in P(A \cap B)$. This gives $P(A) \cap P(B) \subset P(A \cap B)$

Hence $P(A \cap B) = P(A) \cap P(B)$.

Example 32 A market research group conducted a survey of 1000 consumers and reported that 720 consumers like product A and 450 consumers like product B, what is the least number that must have liked both products?

Solution Let U be the set of consumers questioned, S be the set of consumers who liked the product A and T be the set of consumers who like the product B. Given that

$$n(U) = 1000,\ n(S) = 720,\ n(T) = 450$$

So $\qquad n(S \cup T) = n(S) + n(T) - n(S \cap T)$

$$= 720 + 450 - n(S \cap T) = 1170 - n(S \cap T)$$

Therefore, $n(S \cup T)$ is maximum when $n(S \cap T)$ is least. But $S \cup T \subset U$ implies $n(S \cup T) \le n(U) = 1000$. So, maximum values of $n(S \cup T)$ is 1000. Thus, the least value of $n(S \cap T)$ is 170. Hence, the least number of consumers who liked both products is 170.

Example 33 Out of 500 car owners investigated, 400 owned car A and 200 owned car B, 50 owned both A and B cars. Is this data correct?

Solution Let U be the set of car owners investigated, M be the set of persons who owned car A and S be the set of persons who owned car B.

Given that $\qquad n(U) = 500$, $n(M) = 400$, $n(S) = 200$ and $n(S \cap M) = 50$.

Then $\quad n(S \cup M) = n(S) + n(M) - n(S \cap M) = 200 + 400 - 50 = 550$

But $S \cup M \subset U$ implies $n(S \cup M) \le n(U)$.

This is a contradiction. So, the given data is incorrect.

Example 34 A college warded 38 medals in football, 15 in basketball and 20 in cricket. If these medals went to a total of 58 men and only three men got medals in all the three sports, how many received medals in exactly two of the three sports ?