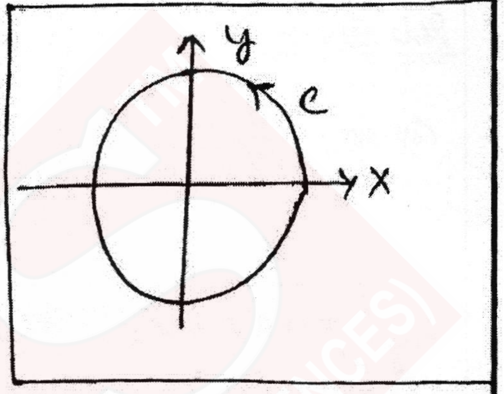
2(c) → If α, β, γ are real numbers such that $\alpha^2 > \beta^2+\gamma^2$. Show that :

$$\int_0^{2\pi} \frac{d\theta}{\alpha+\beta\cos\theta+\gamma\sin\theta} = \frac{2\pi}{\sqrt{\alpha^2-\beta^2-\gamma^2}}$$

Solution:-



Let $z = e^{i\theta}$, then

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$$

$$dz = i\,e^{i\theta}\,d\theta$$

$$\boxed{dz = iz\,d\theta}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{\alpha+\beta\cos\theta+\gamma\sin\theta} = \oint_c \frac{dz}{iz\left[\alpha+\beta\left(\frac{z+z^{-1}}{2}\right)+\gamma\left[\frac{z-z^{-1}}{2i}\right]\right]}$$

$$= \oint_c \frac{2i\,dz}{iz\left[2\alpha i+\frac{2i\beta}{2}(z+z^{-1})+\gamma(z-z^{-1})\right]}$$

$$= \oint_c \frac{2\,dz}{2\alpha zi+z\left[\frac{z^2\beta i+\beta i}{z}+\frac{\gamma(z^2-1)}{z}\right]}$$

$$= \oint_c \frac{2\,dz}{2\alpha zi+z^2\beta i+\beta i+\gamma z^2-\gamma}$$

$$= \oint_c \frac{2\,dz}{z^2(\beta i+\gamma)+2\alpha zi+(\beta i-\gamma)}$$

$$= \oint_c \frac{2\,dz}{(\gamma+i\beta)\left[z^2+\left(\frac{2\alpha i}{\gamma+\beta i}\right)z+\left[\frac{\beta i-\gamma}{\gamma+\beta i}\right]\right]}$$