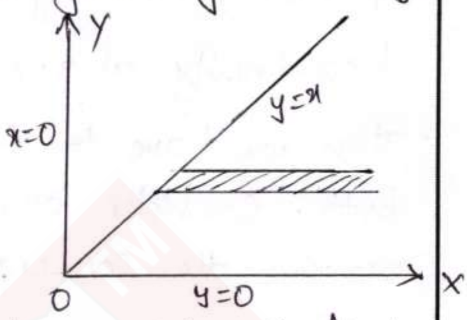
7(c) Evaluate the integral $\int_0^\infty \int_0^x x e^{-x^2/y}\, dx\, dy$ by changing the order of integration.

Sol'n: The limits of integration are given by the straight line $y=x$, $y=0$, $x=0$ and $x=\infty$.

i.e., the region of integration is bounded $y=0$, $y=x$ and infinite boundary.



Hence taking the strips parallel to x-axis, the limits for y are from 0 to ∞, and the limits for x are from $x=y$ to $x=\infty$.

Hence changing the order of integration, we are

$$\int_0^\infty \int_0^x x e^{-x^2/y}\, dy\, dx = \int_{y=0}^\infty \int_{x=y}^\infty x e^{-x^2/y}\, dx\, dy$$

$$= -\int_{y=0}^\infty \int_{-y}^{y} \frac{y}{2} e^t\, dt\, dy \qquad \frac{-x^2}{y} = t$$

$$\frac{-2x\,dx}{y} = dt$$

$$= -\int_0^\infty \frac{y}{2} \left[e^t\right]_{-y}^{\infty} dy \qquad x\,dx = \frac{-y}{2}\, dt$$

where $x=y$

$$= -\int_0^\infty \frac{y}{2}\left[0 - e^{-y}\right] dy \qquad \frac{-y^2}{y} = t \Rightarrow t$$

$$= \int_0^\infty \frac{y}{2} e^{-y}\, dy = \frac{1}{2}\int_0^\infty y e^{-y}\, dy$$

$$= \frac{1}{2}\left[-y(e^{-y}) - \int_0^\infty e^{-y}(-dy)\right]_0^\infty$$

$$= \frac{1}{2}\left[-y e^{-y} + \int_0^\infty e^{-y}\, dy\right]_0^\infty$$

$$= \frac{1}{2}\left[-y e^{-y} - e^{-y}\right]_0^\infty = \frac{1}{2}\left[0 - (0-1)\right]$$

$$= \frac{1}{2}$$