3(a) Show that $\frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2}$, if $0<u<v$

and deduce that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$.

Soln: let $f(x) = \tan^{-1}x$, then for $u<x<v$

$$f'(x) = \frac{1}{1+x^2}$$

Applying the mean value theorem to f, we got

$$\frac{\tan^{-1}v - \tan^{-1}u}{v-u} = \frac{1}{1+\xi^2}, \text{ for } u<\xi<v$$

$$\xi > u \Rightarrow \frac{1}{1+\xi^2} < \frac{1}{1+u^2}$$

$$\xi < v \Rightarrow \frac{1}{1+\xi^2} > \frac{1}{1+v^2}$$

$$\frac{1}{1+v^2} < \frac{\tan^{-1}v - \tan^{-1}u}{v-u} < \frac{1}{1+u^2}$$

$$\frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2}$$

The other result follows by taking $v = 4/3$ and $u=1$

$$\frac{\frac{4}{3}-1}{1+\left(\frac{4}{3}\right)^2} < \tan^{-1}(4/3) - \tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+1}$$

$$\frac{\frac{1}{3}}{25/9} < \tan^{-1}(4/3) - \frac{\pi}{4} < \frac{\frac{1}{3}}{\frac{2}{1}}$$