Show also that the equation of the system cutting all the circles of the coaxal system orthogonally is

(i) $(x^2+y^2)(g+\lambda f)+c(x+\lambda y)=0.$

Origin is a limiting point. Hence the equation of the point circle is

$$x^2+y^2=0.$$

This point circle is also a member of the coaxal system.

Any circle coaxal with the two given circles is

$$x^2+y^2+2gx+2fy+c+k(x^2+y^2)=0.$$

$$x^2+y^2+\frac{2gx}{1+k}+\frac{2fy}{1+k}+\frac{c}{1+k}=0.$$

Centre of this circle is $\left(\frac{-g}{1+k},\frac{-f}{1+k}\right)$ and its radius is

$$\sqrt{\frac{g^2}{(1+k)^2}+\frac{f^2}{(1+k)^2}-\frac{c}{1+k}}$$

If this circle is a limiting point, its radius must be zero

$$\therefore\ g^2+f^2-c(1+k)=0.$$

$$\therefore\ k=\frac{g^2+f^2-c}{c}$$

The other limiting point is $\left[\frac{-g}{1+\frac{g^2+f^2-c}{c}},\ \frac{-f}{1+\frac{g^2+f^2-c}{c}}\right]$

i.e., $\left[\frac{-gc}{g^2+f^2},\ \frac{-fc}{g^2+f^2}\right]$

(ii) Any circle of this system is

$$x^2+y^2+\frac{2gx}{1+k}+\frac{2fy}{1+k}+\frac{c}{1+k}=0 \quad \ldots (1)$$

Let the equation of the circle cutting this system orthogonally be

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \quad \ldots (2)$$

$$\therefore\ \frac{2gg_1}{1+k}+\frac{2ff_1}{1+k}=\frac{c}{1+k}+c_1 \quad \ldots (3)$$

The circle (2) passes through limiting points of the coaxal system. Since one of the limiting point is (0, 0), $c_1=0$.

$\therefore$ The condition (3) becomes

$$\frac{2gg_1}{1+k}+\frac{2ff_1}{1+k}=\frac{c}{1+k}$$

i.e., $2gg_1+2ff_1=c$

$$2g_1\left[g+f\frac{f_1}{g_1}\right]=c$$

Put $\frac{f_1}{g_1}=\lambda$ $\therefore$ $2g_1=\frac{c}{g+\lambda f}$ and $2f_1=2g_1\frac{f_1}{g_1}$

$$=\frac{c\lambda}{g+\lambda f}$$

The equation of the orthogonal system will be

$$x^2+y^2+\left(\frac{c}{g+\lambda f}\right)x+\left(\frac{c\lambda}{g+\lambda f}\right)y=0$$

$$(x^2+y^2)(g+\lambda f)+c(x+\lambda y)=0.$$

It can be verified that this circle passes through the other limiting point as well.

PROBLEMS

1. Prove that from a point (a,b) of the circle $x(x-a)+y(y-b)=0$, two chords, each bisected by the axis of x, can be drawn if a^2 be greater than $8b^2$.

2. A variable chord of the circle $x^2+y^2+2gx+2fy+2c=0$ subtends a right angle at the origin. Show that the locus of the foot of the perpendicular from the origin to the chord is the circle

$$x^2+y^2+gx+fy+c=0.$$

3. Show that the circle on the line joining the points of intersection of the line $lx+my=1$ with the lines $ax^2+2hxy+by^2=0$ as diameter is

$$(am^2-2hlm+bl^2)(x^2+y^2)+2x(hm-bl)+2y(hl-am)+a+b=0.$$

Deduce the condition for the lines $ax^2+2hxy+by^2=0$ to be at right angles.

4. Find the circumcentre of the triangle formed by the lines $x+y=0$, $x-y=0$ and $lx+my=1$. If l and m vary subject to the condition $l^2+m^2=1$, show that the locus of its circumcentre is the curve $(x^2-y^2)^2=x^2+y^2$.

5. Prove that the polar with respect to the circle $x^2+y^2=c^2$ of any point on the circle $(x+a)^2+(y-b)^2 = k^2$ always touches the curve $(ax+by+c)^2 = k^2(x^2+y^2)$.

6. Show that the locus of the middle points of chords of the circle $x^2+y^2=a^2$ which subtend a right angle at the point (h,k) is a circle whose centre is

$$\left(\frac{h}{2},\frac{k}{2}\right)$$